

Infrared Consistency of NSVZ and DRED Supersymmetric Gluodynamics

V. Elias

Department of Applied Mathematics
University of Western Ontario
London, Ontario N6A 5B7
Canada

Abstract

Padé approximant methods are applied to the known terms of the DRED β -function for $N = 1$ supersymmetric SU(3) Yang-Mills theory. Each of the $[N|M]$ approximants with $N + M \leq 4$, $M \neq 0$ constructed from this series exhibits a positive pole which precedes any zeros of the approximant, consistent with the same infrared-attractor pole behaviour known to characterize the exact NSVZ β -function. A similar Padé-approximant analysis of truncations of the NSVZ series is shown consistently to reproduce the geometric-series pole of the exact NSVZ β -function.

1. Introduction: Supersymmetric Gluodynamics and the Adler-Bardeen Theorem

Nearly two decades ago, Novikov, Shifman, Vainshtein, and Zakharov (NSVZ) derived the exact β -function for supersymmetric gauge theories with no matter fields [1]. However for the $N = 1$ case, it was recognized early-on that scheme dependence characterizes higher-than-three loop terms of this β -function; a prior three-loop calculation using now-standard dimensional reduction (DRED) methods [2] was seen to yield a β -function that differs from the NSVZ result (as anticipated in [3]) at three-loop-order. The present work seeks to address whether this discrepancy is indicative of differences more profound than those which can be accommodated by order-by-order redefinition of the coupling constant.

For simplicity, we will focus our attention on $N = 1$ supersymmetric SU(3) Yang-Mills theory (supersymmetric gluodynamics). Arguments advanced by Jones [3], which are recapitulated below, can be used to extract

the exact NSVZ β -function for this theory. Within supersymmetric gluodynamics, the supermultiplet structure of the anomalies entails the following relation:

$$\partial_\mu J_5^\mu(x) = \partial_\mu \left[\frac{1}{2} \bar{\lambda}(x) \gamma^\mu \gamma_5 \lambda(x) \right] = \frac{\beta(g)}{g} \cdot \frac{4}{3} G(x) \quad (1.1)$$

$[\beta(g) = \mu dg/d\mu]$ where $\lambda(x)$ are gluino fields, and where

$$G = -\frac{1}{8} F \tilde{F} + \frac{1}{4} \partial_\mu \left[\bar{\lambda}(x) \gamma^\mu \gamma_5 \lambda(x) \right] \quad (1.2)$$

is a component of the Wess-Zumino chiral supermultiplet (A,B, ψ ,F,G). Eq. (1.2) may be rearranged to read

$$F \tilde{F} = -8 \left[G - \frac{1}{2} \partial_\mu J_5^\mu \right]. \quad (1.3)$$

However, the Adler-Bardeen theorem requires for $N_c = 3$ that [4]

$$\partial_\mu J_5^\mu = \frac{1}{2} \frac{3g^2}{16\pi^2} F \tilde{F}. \quad (1.4)$$

By substituting (1.3) into the right-hand side of the (1.4), one finds that

$$\partial_\mu J_5^\mu = -\frac{3g^2}{4\pi^2} \left[G - \frac{1}{2} \partial_\mu J_5^\mu \right] \quad (1.5)$$

in which case

$$\partial_\mu J_5^\mu(x) = \left[(-3g^2/4\pi^2)/(1 - 3g^2/8\pi^2) \right] G(x). \quad (1.6)$$

Comparing the right-hand sides of (1.1) and (1.6), one finds that

$$\beta(g)/g = -\frac{9g^2/16\pi^2}{1 - 3g^2/8\pi^2}. \quad (1.7)$$

This β -function manifestly acquires a pole at $\alpha = 2\pi/3$, and indeed corresponds to the NSVZ β -function whose infrared properties are discussed by Kogan and Shifman in ref. [5]. In other words, the exact NSVZ β -function for supersymmetric gluodynamics can be extracted by imposing the Adler-Bardeen theorem on the supermultiplet structure of the anomalies [3].

Alternatively, the analysis of ref. [3] can be turned around to show that the NSVZ β -function for supersymmetric gluodynamics, as derived in [1] via instanton calculus methods, is consistent to all orders with the Adler-Bardeen theorem. By contrast, the DRED β -function, which differs from the NSVZ β -function subsequent to identical leading- and next-to-leading-order contributions, is necessarily *inconsistent* with the Adler-Bardeen theorem after next-to-leading order.

The question of whether or not the DRED β -function is therefore “wrong” can be sidestepped by a perturbative redefinition of the coupling [6, 7]. Denoting by x and y the corresponding NSVZ- and DRED-coupling parameters ($x \equiv g^2/16\pi^2$), the respective perturbative β -functions taken to three non-leading orders [7] are consistent with the coupling constant redefinition ¹

$$y = x \left[1 + 27x^2 + 351x^3 + \dots \right]. \quad (1.8)$$

However, the deeper question of validity ultimately rests upon the consistency of the non-perturbative content of the DRED and NSVZ schemes. In ref. [6], for example, the suggestion is made that a relation may be found between possible infrared fixed points of β -functions in both schemes when matter fields are present. Our focus here, though, will be on β -function properties in the *absence* of matter fields, since these properties are fully accessible for the NSVZ case.

In the NSVZ scheme, the β -function (1.7) leads to a double-valued couplant. The pole at $\alpha = 2\pi/3$ ($x = 1/6$), is an infrared-attractor point common to both phases of the couplant [5], *i.e.*, an asymptotically-free phase ($x < 1/6$ and $x \rightarrow 0$ as $\mu \rightarrow \infty$) and a non-asymptotically free strong phase (x *increases* monotonically with μ and is greater than $1/6$). Thus, the infrared content of supersymmetric gluodynamics is governed by a *pole*, as opposed to an infrared fixed point corresponding to a positive β -function zero.

It would clearly be of value to determine whether the same infrared behaviour characterizes the DRED β -function for supersymmetric gluodynamics. However, the DRED β -function series is presently known only to four-loop order. Consequently, we choose to utilize Padé-approximant methods that are already known to be of value in extracting poles of functions

¹In the notation of ref. [7], $(g^{DRED})^2 = (g^{NSVZ})^2 + \delta^{(1)} + \delta^{(2)} + \delta^{(3)}$. If $n_f = 0$ and $N_c = 3$, the ref. [7] group-theoretical factors for supersymmetric gluodynamics are $C(R) = T(R) = 0$, $C(G) = 3$, and $Q = -9$, in which case $\delta^{(1)} = 0$, $\delta^{(2)} = 27x^2(g^{NSVZ})^2$, and $\delta^{(3)} = 351x^3(g^{NSVZ})^2$.

underlying truncated power series [8]. In Section 2, we test this approach by examining whether Padé-approximants constructed from truncations of the geometric NSVZ β -function series succeed in reproducing the infrared-attractor pole characterizing the full β -function's infrared content. We find this to be the case for all $M \neq 0$ $[N|M]$ approximants that can be constructed from the first five terms of the NSVZ β -function series (the $M = 0$ case is the truncated series itself). We also find that asymptotic Padé-approximant-prediction (APAP) formulae for subsequent terms of a truncated geometric series (*e.g.*, truncations of the NSVZ β -function series) are fully consistent with that series remaining geometric to next order.

In Section 3, the same Padé-approximant procedures are applied to known terms of the DRED β -function series. We reconfirm the observation [7] that APAP methods are successful (within 10% accuracy) in predicting the four-loop term of this series from its first three terms. We find that all $[N|M]$ approximants constructed from the four (or fewer) known terms of the DRED β -function series ($N + M \leq 3$) are characterized by a positive pole which precedes any positive zeros. Moreover, if these four known terms are augmented by the APAP prediction for the five-loop term of the series, this same result is found to be applicable for the $N + M = 4$ case of $[N|M]$ approximants to the five-loop series.

Such a pole common to *all* $N + M \leq 4$ approximants to the DRED β -function is suggestive of an infrared-attractor pole for the true DRED β -function. However, the magnitude of the pole, as indicated by the full set of $N + M \leq 4$ approximants, is consistently seen to be less than the $x = 1/6$ value characterizing the NSVZ β -function. This contradicts expectations [*e.g.*, eq. (1.8)] that couplant values in the DRED scheme should be *larger* than their corresponding values in the NSVZ scheme. In Section 4, this issue is examined in detail. In particular, the formula (1.8) is demonstrated to be a special case of a more general formula which *does* permit DRED couplant values that are smaller than their corresponding NSVZ values for the asymptotically-free phase. We thus conclude that the smaller infrared-attractor pole that appears to characterize the DRED β -function is not necessarily incompatible with the pole characterizing the NSVZ β -function for supersymmetric gluodynamics.

2. Padé-Approximants and NSVZ Supersymmetric Gluodynamics

We would like to probe the infrared structure of the DRED β -function for supersymmetric gluodynamics, and compare its infrared properties to those of the NSVZ β -function. The NSVZ β -function for supersymmetric gluodynamics ($N = 1$ supersymmetric $SU(3)$ Yang-Mills theory) is

$$\mu^2 \frac{dx}{d\mu^2} \equiv \beta^{NSVZ}(x) = -9x^2/(1 - 6x), \quad (2.1)$$

where

$$x \equiv \alpha^{NSVZ}(\mu)/4\pi = (g^{NSVZ}(\mu))^2/16\pi^2. \quad (2.2)$$

This β -function's infrared behaviour is governed by a pole at $x = 1/6$. Values of the couplant between zero, the ultraviolet fixed point, and $1/6$, the infrared-attractor pole, correspond to the asymptotically-free phase of supersymmetric gluodynamics, as discussed in [5]. The perturbative series for the NSVZ β -function is clearly geometric to all orders:

$$\beta^{NSVZ}(x) = -9x^2 \left[1 + 6x + 36x^2 + 216x^3 + \dots \right]. \quad (2.3)$$

By contrast the corresponding DRED β -function is known only from the first four terms of its perturbative series [2, 7]:

$$\mu^2 \frac{dy}{d\mu^2} \equiv \beta^{DRED}(y) = -9y^2 \left[1 + 6y + 63y^2 + 918y^3 + \dots \right], \quad (2.4)$$

$$y \equiv \alpha^{DRED}(\mu)/4\pi = (g^{DRED}(\mu))^2/16\pi^2. \quad (2.5)$$

In the absence of any further information about the DRED β -function, we necessarily have to make use of Padé-approximant methods to explore whether the same infrared behaviour characterises the asymptotically-free phases of DRED and NSVZ versions of supersymmetric gluodynamics.

The general problem of utilizing Padé approximants to ascertain infrared properties of β -functions from their perturbative series

$$\beta(z) = \beta_0 z^k \left[1 + R_1 z + R_2 z^2 + R_3 z^3 + R_4 z^4 + \dots \right] \quad (2.6)$$

is addressed in reference [9], and is an example of the use of Padé-approximant poles to probe the singularity structure of functions underlying the (known)

terms of truncated series [8]. One self-evident criterion for such use of Padé approximants is the requirement that Padé-approximant predictions of R_{N+1} (based on known values for $\{R_1, R_2, \dots, R_N\}$) become progressively more accurate as N increases. If such is the case, and if a leading zero (infrared fixed point) or pole (infrared-attractor point) *consistently* emerges from all possible approximants whose power series reproduce the known coefficients R_k , then that leading structure (be it a zero or a pole) is unlikely to be a Padé artifact (*e.g.*, a defect pole) such as might arise from only a single Padé approximant to the truncated series [8].

Padé approximant predictions of R_{N+1} are based on $[N - M|M]$ Padé approximants whose Maclaurin expansions reproduce the “known” terms $1 + R_1 z + R_2 z^2 + \dots + R_N z^N$ of the series. For example, even two known terms $1 + R_1 z$ are sufficient to determine the $[0|1]$ Padé approximant $1/(1 - R_1 z) = 1 + R_1 z + R_1^2 z^2 + \dots$, which can then be said to predict the “unknown” next-order coefficient R_2 : $R_2^{[0|1]} = R_1^2$. If this prediction differs in sign or greatly in magnitude from the true calculated value of R_2 , then there is little reason for believing that the true series sum exhibits the same pole at $z = 1/R_1$ that characterises the $[0|1]$ approximant. On the other hand, if estimated values of R_{N+1} from $[N - M|M]$ Padé approximants grow progressively more accurate with increasing N , then properties (such as the whether the first positive pole precedes the first positive zero) shared by such approximants are likely to characterise the series itself. Of particular interest are the “APAP” algorithms [10]

$$R_3^{APAP} = 2R_2^3 / [R_1^3 + R_1 R_2], \quad (2.7)$$

$$R_4^{APAP} = \frac{\{R_3^2(R_2^3 + R_1 R_2 R_3 - 2R_1^3 R_3)\}}{\{R_2(2R_2^3 - R_1^3 R_3 - R_1^2 R_2^2)\}}, \quad (2.8)$$

obtained from assuming that $[N|1]$ approximants grow progressively accurate in predicting R_{N+2} as N increases: ²

$$\frac{R_{N+2}^{[N|1]} - R_{N+2}^{true}}{R_{N+2}^{true}} = -\frac{k_1}{N + 1 + k_2/k_1} = -\frac{k_1}{N + 1} + \frac{k_2}{(N + 1)^2} + \dots \quad (2.9)$$

²Eq. (2.9) is the $M = 1$ case of the ref. [11] formula $\frac{R_{N+M+1}^{[N|M]} - R_{N+M+1}^{true}}{R_{N+M+1}^{true}} = -\frac{M!A^M}{[N+M+aM+b]^M}$ with $k_1 = -A$ and $k_2 = -A(a+b)$. The formula (2.7) is explicitly derived in ref. [12] via $[0|1]$ and $[1|1]$ approximants from (2.9), with the subleading error coefficient k_2 chosen to be zero, as in [13]. The formula (2.8) is derived in ref. [10] via $[0|1]$, $[1|1]$, and $[2|1]$ approximants, and is equivalent to the prediction procedure described in [11].

In (2.9), k_1 and k_2 are leading and next-to-leading constants determined empirically from the error in $[N-1|1]$ and $[N-2|1]$ approximants in predicting “known” series coefficients R_{N+1} and R_N , respectively.

The formulae (2.7) and (2.8) are fully consistent with geometric series, such as the one characterising the perturbative NSVZ β -function (2.3). If $R_1 = r$ and $R_2 = r^2$, it is evident from (2.7) that $R_3^{APAP} = r^3$. The situation is slightly more subtle for (2.8); if $R_1 = r$, $R_2 = r^2$, and $R_3 = r^3$, R_4^{APAP} is indeterminate. However, if R_1 , R_2 , and R_3 differ infinitesimally from geometric series values; *i.e.*, if

$$R_1 = r(1 + \epsilon_1), \quad R_2 = r^2(1 + \epsilon_2), \quad R_3 = r^3(1 + \epsilon_3), \quad (2.10)$$

one easily sees from (2.8) that $R_4^{APAP} \rightarrow r^4$ as $\epsilon_{1,2,3} \rightarrow 0$:

$$\begin{aligned} R_4^{APAP} &= \frac{\{r^6(1 + \epsilon_3)^2[-5\epsilon_1 + 4\epsilon_2 - \epsilon_3 + \mathcal{O}(\epsilon_k^2)]\}}{\{r^2(1 + \epsilon_2)[-5\epsilon_1 + 4\epsilon_2 - \epsilon_3 + \mathcal{O}(\epsilon_k^2)]\}} \\ &= r^4(1 + 2\epsilon_3 - \epsilon_2 + \mathcal{O}(\epsilon_k^2)). \end{aligned} \quad (2.11)$$

Thus the APAP algorithms (2.7) and (2.8) anticipate the geometric series evident from the first three or four known terms of (2.3) for the NSVZ β -function, and would therefore lead one to anticipate for that series a geometric-series pole at $z = 1/6$. This is born out by the $[2|1]$, $[1|2]$, and $[0|3]$ approximants to the “known” series terms $S = 1 + R_1 z + R_2 z^2 + R_3 z^3$ whose coefficients $\{R_1, R_2, R_3\}$ are given by (2.10):

$$S^{[2|1]} = \frac{1 + zr[\epsilon_1 - \epsilon_3 + \mathcal{O}(\epsilon_k^2)] + z^2 r^2[\epsilon_2 - \epsilon_1 - \epsilon_3 + \mathcal{O}(\epsilon_k^2)]}{1 - zr[1 + \epsilon_3 - \epsilon_2 + \mathcal{O}(\epsilon_k^2)]}, \quad (2.12)$$

$$\begin{aligned} S^{[1|2]} &= \frac{\{1 + zr[\epsilon_1 - 2\epsilon_2 + \epsilon_3 + \mathcal{O}(\epsilon_k^2)] / [2\epsilon_1 - \epsilon_2 + \mathcal{O}(\epsilon_k^2)]\}}{\left\{1 + zr \frac{[\epsilon_3 - \epsilon_1 - \epsilon_2 + \mathcal{O}(\epsilon_k^2)]}{[2\epsilon_1 - \epsilon_2 + \mathcal{O}(\epsilon_k^2)]} + z^2 r^2 \frac{[2\epsilon_2 - \epsilon_1 - \epsilon_3 + \mathcal{O}(\epsilon_k^2)]}{[2\epsilon_1 - \epsilon_2 + \mathcal{O}(\epsilon_k^2)]}\right\}} \\ &= \frac{\{\epsilon_1(2 + rz) - \epsilon_2(1 + 2rz) + \epsilon_3 rz + \mathcal{O}(\epsilon_k^2)\}}{\{\epsilon_1(2 - rz - r^2 z^2) - \epsilon_2(1 + rz - 2r^2 z^2) + \epsilon_3(rz - r^2 z^2) + \mathcal{O}(\epsilon_k^2)\}}, \end{aligned} \quad (2.13)$$

$$S^{[0|3]} = \frac{1}{1 - rz(1 + \epsilon_1) + r^2 z^2(2\epsilon_1 - \epsilon_2) + r^3 z^3(-2\epsilon_1 + \epsilon_2 - \epsilon_3) + \mathcal{O}(\epsilon_k^2)} \quad (2.14)$$

All three expressions reduce to $1/(1-rz)$ in the limit $\epsilon_{1,2,3} \rightarrow 0$. Moreover, if (2.10) is augmented to include $R_4 = r^4(1 + \epsilon_4)$, as anticipated by (2.11), one can similarly show via formulae from Section 2 of [9] that $S^{[3|1]}$, $S^{[2|2]}$, $S^{[1|3]}$ and $S^{[0|4]}$ determined from the “known” series terms $1 + R_1z + R_2z^2 + R_3z^3 + R_4z^4$ all reduce to $1/(1-rz)$ in the limit $\epsilon_{1,2,3,4} \rightarrow 0$, thereby reproducing the single pole behaviour of an underlying geometric series, such as the $r = 6$ geometric series characterizing the NSVZ β -function (2.1).

It should be noted that this $1/(1-rz)$ behaviour also characterizes the $\epsilon_k \rightarrow 0$ limit of the lower-approximant expressions $S^{[0|1]}$, $S^{[1|1]}$ and $S^{[0|2]}$ [as obtained from only $R_1 = r(1 + \epsilon_1)$ and $R_2 = r^2(1 + \epsilon_2)$]. Thus for $N + M \leq 4$ and $M \neq 0$ [the trivial $M = 0$ case corresponds to the truncated series itself], *all* $[N|M]$ approximants constructed from the first $N + M + 1$ terms of a truncated series $\sum_{k=0}^{N+M} R_k z^k$ reduce ultimately to the geometric series sum $1/(1-rz)$ as $R_k \rightarrow r^k$ ($k \leq N + M$).³ Note also that any other departures from a single pole at $z = 1/r$ (*e.g.*, additional zeros or poles) are seen to vanish from these $[N|M]$ approximants as $R_k \rightarrow r^k$ for $k \leq N + M$.

These conclusions are clearly applicable to the set of $[N + M]$ approximants ($N + M \leq 4$) constructed from truncations of the NSVZ β -function series (a geometric series with $r = 6$), and demonstrate the validity of such approximants in predicting the pole at $1/6$ arising from the infinite series sum, as well as in *not* predicting further spurious poles or zeros. In the section that follows, we will utilize the same ($N + M \leq 4$) set of Padé approximants to the DRED β -function series for supersymmetric gluodynamics in order to identify any common features (zeros or poles) that may be indicative of the true infrared content of the DRED β -function series sum.

3. Padé-Approximants and DRED Supersymmetric Gluodynamics

The DRED β -function (2.4) is not a geometric series. The application of Padé-approximant methods to that β -function is predicated on the idea that such approximants grow progressively more accurate in predicting next-order terms, as in the error formula (2.9). This behaviour seems to be born out by

³The restriction $N + M \leq 4$ for the degree of the truncated series is likely unnecessary; we have not examined any $N + M > 4$ approximants, but suspect the result stated here to be a general one for the coefficients R_k of any truncated series.

the known coefficients appearing in (2.4). The APAP estimate via (2.7) for R_3 , based on $R_1 = 6$ and $R_2 = 63$, is $R_3^{APAP} = 841.9$,⁴ which is in surprisingly good agreement with $R_3 = 918$, supersymmetric gluodynamics' true R_3 value in the DRED scheme [7]. By contrast, the naive estimate for R_3 based on the $[1|1]$ approximant reproducing $1+6y+63y^2$ is $R_3^{[1|1]} = (63)^2/6 = 661.5$. Thus, the success of (2.7) [which devolves from (2.9)] in predicting R_3 more accurately than the $[1|1]$ approximant itself may be seen as evidence for the progressive increase in accuracy of $[N-M|M]$ approximants in reproducing the DRED β -function as N increases.

The known terms (2.4) are sufficient to determine $[2|1]$ - and $[1|2]$ -approximant versions of the DRED β -function:

$$\beta^{[2|1]}(y) = -9y^2 \left[\frac{1 - 8.5714y - 24.4286y^2}{1 - 14.5714y} \right] \quad (3.1)$$

$$\beta^{[1|2]}(y) = -9y^2 \left[\frac{1 - 14y}{1 - 20y + 57y^2} \right]. \quad (3.2)$$

Both of these approximations to the true DRED β -function necessarily have an ultraviolet fixed point at $y = 0$. However, their infrared behaviour, if extractable at all, should manifest itself in a positive zero or pole common to the asymptotically free phase of both expressions. We find that both approximants are consistent with infrared dynamics governed by an infrared-attractor pole, as is the case for the NSVZ β -function. For $\beta^{[2|1]}$, a pole at $y = 0.06863$ is seen to precede the first positive numerator zero (0.09236). Similarly, the first positive pole of $\beta^{[1|2]}$ is seen to occur at $y = 0.0604$, which precedes the positive numerator zero at $y = 0.07143$. Thus in both cases, the pole controls the infrared dynamics of the asymptotically-free phase in much the same way as the NSVZ β -function pole.⁵ Such a positive pole also characterizes the $[0|3]$ -approximant to the DRED β -function, as determined from the full set of known terms in the series (2.4), as well as $[1|1]$ and $[0|2]$ approximants constructed from the first three terms of that series.

⁴This estimate is virtually the same as that of ref. [13] for the $n_f = 0$, $N_c = 3$ case, following a procedure also based on the error formula (2.9) with $k_2 = 0$. The $\alpha = 2.4$ estimate derived in ref. [13] implies that $\beta_3 = 486(1 + 6\alpha) = 7484$ and that $R_3 \equiv \beta_3/\beta_0 = 832$.

⁵The larger numerator zero, if meaningful at all, corresponds to an ultraviolet fixed point of a subsequent (non-asymptotically free) phase of the couplant. See the discussion of subsequent positive zeros/poles for toy models discussed in section 2 of ref. [9].

One can consider higher Padé-approximants by using (2.8) to obtain an estimate of R_4 . Using $R_1 = 6$, $R_2 = 63$, and $R_3 = 918$, one sees from (2.8) that $R_4^{APAP} = 16,874$, corresponding to $\beta_4^{DRED} = 151,870$.⁶ Given this estimate for R_4 , it is straightforward to generate $[3|1]$, $[2|2]$, and $[1|3]$ Padé-approximant versions of the DRED β -function (formulae for constructing these approximants from series are explicitly given in [9]):

$$\beta^{[3|1]}(y) = -9y^2 \left[\frac{1 - 12.38y - 47.29y^2 - 240.0y^3}{1 - 18.38y} \right] \quad (3.3)$$

$$\beta^{[2|2]}(y) = -9y^2 \left[\frac{1 - 22.21y + 36.93y^2}{1 - 28.21y + 143.2y^2} \right] \quad (3.4)$$

$$\beta^{[1|3]}(y) = -9y^2 \left[\frac{1 - 19.57y}{1 - 25.57y + 90.41y^2 + 150.4y^3} \right]. \quad (3.5)$$

As before, the first positive pole of each of these functions precedes the first positive zero (Table 1), consistent with the pole-dominated infrared dynamics of the NSVZ β -function's asymptotically-free phase.

Table 1 tabulates the first positive pole and zero of *every* Padé-approximant version of the DRED β -function obtainable from known ($R_1 = 6, R_2 = 63, R_3 = 918$) and APAP-estimated ($R_4^{APAP} = 16,874$) values of the leading series coefficients. Every approximant considered exhibits a positive pole which precedes any positive zeros of the approximant. Such an ordering implies that the first positive pole is necessarily associated with the infrared limit of the asymptotically-free phase of the couplant. Indeed, the leading positive Padé-approximant zero can be associated with an infrared fixed-point for the asymptotically-free phase only if it *precedes* any poles of the same approximant [10]. This is clearly *not* the case for DRED β -function, which appears (from all of its approximant versions) to exhibit the same infrared-attractor pole dynamics as the NSVZ β -function for supersymmetric gluodynamics.

⁶This estimate is larger than the estimate $\beta_4 = 113,000$ of [11], because of that latter estimate's explicit use of $\alpha = 2.4$ as an input parameter. If α is taken to be 2.4 instead of its true value of $8/3$, then $R_3 = 832$ (see Footnote 4). Eq. (2.8) would then imply that $R_4^{APAP} = 12,700$ and that $\beta_4 = 9R_4^{APAP} = 114,000$, consistent with the estimate in [11].

4. Discussion

For the asymptotically-free phase of supersymmetric gluodynamics, the existence of an infrared-attractor pole within every Padé-approximant version of the DRED β -function is strong evidence that such a pole is not an artifact. Thus, it appears that DRED and NSVZ lead to infrared dynamics that are qualitatively equivalent. However, it must also be noted from Table 1 that the DRED version of this infrared-attractor pole appears to be *smaller* than $1/6$, the value of the NSVZ pole. [The value of $1/6$ in Table 1 obtained from the $[0|1]$ approximant to the DRED β -function is irrelevant, as the exact same approximant characterizes the true NSVZ β -function, as well as the $[0|1]$ approximant generated by its leading series terms.] There is, of course, no reason why the poles for the two schemes *should* have the same value; it is evident from (1.8) that corresponding values of the couplant in the two schemes are inequivalent.

Nevertheless, if NSVZ and DRED values of the couplant are equal at some (large) finite value of μ , one would anticipate from the β -function series (2.3) and (2.4) that any particular NSVZ couplant value at a lower value of μ scheme should correspond to a *larger* couplant value in the DRED scheme, since the β -function coefficients of the latter are term-by-term larger than the β -function coefficients of the former. This anticipated inequality is reflected in the relation (1.8), which clearly generates DRED couplants (y) that are larger than their corresponding NSVZ couplants (x). Although the value $x = 1/6$ is very likely outside the radius of convergence of the series in (1.8), the above observations seem indicative of a fundamental inconsistency in the infrared content of DRED and NSVZ supersymmetric gluodynamics. Specifically, if the infrared-attractor poles of the two schemes are in correspondence via a redefinition of the couplant from one scheme to another, then (1.8) suggests that the DRED pole should be *larger* than the NSVZ pole.

In fact, it may indeed be possible for a given benchmark value of the NSVZ couplant to correspond to a *smaller* value of the DRED couplant within supersymmetric gluodynamics. For the perturbative regime, one can demonstrate that (1.8) is not necessarily the most general possible relationship between couplants in the asymptotically-free phases of the two schemes, as defined by their respective β -functions. Consider first a general parametrisation of the perturbative relation between the two couplants that is consistent with both couplants being in their asymptotically-free phases [*i.e.*, when

$x \rightarrow 0$, one must also require that $y \rightarrow 0$]:

$$y = x(1 + \alpha x + \beta x^2 + \gamma x^3 + \dots) \quad (4.1)$$

One sees from the series expansion (2.3) of the NSVZ β -function that

$$\begin{aligned} \beta^{DRED}(y) &= \mu^2 \frac{dy}{d\mu^2} = \mu^2 \frac{dx}{d\mu^2} [1 + 2\alpha x + 3\beta x^2 + 4\gamma x^3 + \dots] \\ &= -9x^2 [1 + (6 + 2\alpha)x + (36 + 3\beta + 12\alpha)x^2 \\ &\quad + (216 + 4\gamma + 72\alpha + 18\beta)x^3 + \dots] \end{aligned} \quad (4.2)$$

However, if one substitutes (4.1) into (2.4), the series expansion for the DRED β -function, one finds that

$$\begin{aligned} \beta^{DRED}(y) &= -9x^2 [1 + (6 + 2\alpha)x + (63 + 18\alpha + \alpha^2 + 2\beta)x^2 \\ &\quad + (918 + 252\alpha + 18\alpha^2 + 2\alpha\beta + 18\beta + 2\gamma)x^3 + \dots] \end{aligned} \quad (4.3)$$

Comparing the final line of (4.2) to (4.3), one finds that

$$\beta = 27 + 6\alpha + \alpha^2, \quad \gamma = 351 + 117\alpha + 15\alpha^2 + \alpha^3, \quad (4.4)$$

and that

$$y = x [1 + \alpha x + (27 + 6\alpha + \alpha^2)x^2 + (351 + 117\alpha + 15\alpha^2 + \alpha^3)x^3 + \dots]. \quad (4.5)$$

In the absence of any further input information, the value of the coefficient α in (4.1) remains *undetermined*, a consequence of the fact that DRED and NSVZ β -functions agree to two-loop order. If we choose $\alpha = 0$, we recover the relation (1.8), which implies $y > x$ for corresponding perturbative couplant values in DRED and NSVZ, respectively. The choice $\alpha = 0$ is intuitive, in that it forces corresponding NSVZ and DRED couplants to agree to two orders, as is the case for the β -functions themselves of both schemes. However, it appears that the only methodological constraint one is forced to impose is that both couplants be in their asymptotically free phases, as remarked above, and the form of (4.1) is consistent with this requirement for *any* finite value of α . We see that if α is sufficiently negative, the terms in the series (4.5) may alternate in sign. For example, if we choose $\alpha = -6$, we see from the four terms of the series (4.5) that the NSVZ pole at $x = 1/6$ corresponds

to a DRED value $y = 5/48$. If (for this value of α) the series continues term-by-term to decrease in magnitude and alternate in sign, the three- and four-term partial-sum values $1/8$ and $5/48$ become upper and lower bounds on the DRED value corresponding to the NSVZ pole at $x = 1/6$.

The above scenario is, of course, a contrived one. The point we wish to make, however, is that there appears to be no real restriction on the relative sizes of corresponding DRED and NSVZ values of the couplant within supersymmetric gluodynamics in the perturbative region. Thus, we conclude that

- 1) the Padé-approximant analysis indicates the existence of an infrared-attractor pole within DRED supersymmetric gluodynamics that is smaller than the infrared-attractor pole already known to characterise NSVZ supersymmetric gluodynamics [5], and that
- 2) the relative sizes of these poles are *not necessarily* incompatible with the order-by-order correspondence between couplants of the two schemes.

Acknowledgments

I am indebted to V. A. Miransky for suggesting a Padé-approximant discussion of NSVZ supersymmetric gluodynamics, and to F. A. Chishtie and T. G. Steele for numerous discussions related to the material presented here. I am also grateful for two weeks of fruitful interaction on Padé-approximant methods with Mark Samuel immediately prior to his untimely passing. Research support from the Natural Sciences and Engineering Research Council of Canada is also gratefully acknowledged.

References

- [1] Novikov V, Shifman M, Vainshtein A and Zakharov V, 1983 *Nucl. Phys.* B **229** 381
- [2] Avdeev L N, Chochia G A and Vladimirov A A 1981 *Phys. Lett.* B **105** 272
- [3] Jones D R T 1983 *Phys. Lett.* B **123** 45

- [4] Jones D R T and Leveille J 1982 *Nucl. Phys. B* **206** 473
- [5] Kogan I I and Shifman M 1995 *Phys. Rev. Lett.* **75** 2085
- [6] Jack I, Jones D R T and North C G 1997 *Nucl. Phys. B* **486** 479
- [7] Jack I, Jones D R T and Pickering A 1998 *Phys. Lett. B* **435** 61
- [8] Baker G and Graves-Morris P 1981 *Padé-Approximants* [Vol. 13 of *Encyclopedia of Mathematics and its Applications*] (Reading, MA: Addison-Wesley) pp 48-57
- [9] Chishtie F A, Elias V, Miransky V A and Steele T G 2000 *Prog. Theor. Phys.* **104** (to appear: Los Alamos Archive Preprint hep-ph/9905291).
- [10] Elias V, Steele T G, Chishtie F, Migneron R and Sprague K 1998 *Phys. Rev. D* **58** 116007
- [11] Ellis J, Jack I, Jones D R T, Karliner M and Samuel M A 1998 *Phys. Rev. D* **57** 2665
- [12] Ahmady M R, Chishtie F A, Elias V and Steele T G 2000 *Phys. Lett. B* **479** 201
- [13] Jack I, Jones D R T and Samuel M A 1997 *Phys. Lett. B* **407** 143

Approximant	Inputs	First Positive Pole	First Positive Zero
[0 1]	R_1	1/6	—
[1 1]	$R_{1,2}$	2/21	2/9
[0 2]	$R_{1,2}$	1/9	—
[2 1]	$R_{1,2,3}$	0.0686	0.0924
[1 2]	$R_{1,2,3}$	0.0604	0.0714
[0 3]	$R_{1,2,3}$	0.0773	—
[3 1]	$R_{1,2,3,4}$	0.0544	0.0617
[2 2]	$R_{1,2,3,4}$	0.0464	0.0490
[1 3]	$R_{1,2,3,4}$	0.0479	0.0511
[0 4]	$R_{1,2,3,4}$	0.0745	—

Table 1: The first positive pole and first positive zero of all possible Padé approximant versions of the DRED β -function are tabulated for $R_1 = 6$, $R_2 = 63$, $R_3 = 918$, and $R_4 = 16,874$, as discussed in the text. Series parameters used to determine each approximant are listed in the second column. Every approximant exhibits a positive pole which precedes any positive zeros of the same approximant.